Solutions to Comprehensive: Numerical Analysis (30 points). Autumn 1993

Problem I (16 points). **Iterative Solution of Linear Equations**

This question concerns the solution of systems of linear equations in the form

$$Ax = b, \tag{1}$$

where A is a $m \times m$ matrix and x and b are vectors of length m.

(a) Describe the simplest form of *iterative improvement* (also known as *residual correction*) to solve (1). Describe, and briefly explain, the effect of machine precision on this algorithm.

SOLUTION:

Given an approximation x^m to the solution x of (1), the residual r^m is defined by

$$r^m = b - Ax^m.$$

Iterative improvement generates the sequence of approximations

$$x^{m+1} = x^m + \hat{e}^m$$

where $\hat{e}$ is the computed solution of

$$Ae^m = r^m.$$

For such an iteration it is important to obtain accurate values for r^m relative to the precision used in the remainder of the calculation. The reason is simply that otherwise the errors in $\hat{e}^m$ may be comporable with the errors in the original calculation.

(b) Given a matrix C which approximates the inverse of A, consider the following general residual correction method for the solution of (1):

$$r^m = b - Ax^m,$$

$$x^{m+1} = x^m + Cr^m.$$

State the precise condition under which this iteration converges; prove your assertion.

SOLUTION: *The iteration converges provided that the spectral radius of the matrix $I - CA$ is less than one.* To prove this note that the iteration gives

$$x^{m+1} - x = x^m - x + C[b - Ax^m] = x^m - x + C[Ax - Ax^m].$$

Hence the error $\delta^m = x^m - x$ satisfies

$$\delta^{m+1} = [I - CA]\delta^m.$$

It is well-known that it is necessary and sufficient for the spectral radius of $I - CA$ to be less than one for the iterates δ^m to converge to zero, independent of the choice of initial vector.

(c) Given that A may be written as $A = L + D + U$ where L and U are lower and upper triangular respectively and D is diagonal, define the Jacobi and Gauss-Siedel iterations for the solution of (1).

SOLUTION: We have

$$(L + D + U)x = b.$$

The *Jacobi iteration* is to generate x^m according to

$$Dx^{m+1} = b - [L + U]x^m.$$

This also shows that $x^{m+1} \in [a, b]$ and the induction is complete. Letting $m \to \infty$ gives the desired result.

(c) Using part (b) show that Newton iteration converges to a root α of (2) provided that f is twice continuously differentiable and $f'(\alpha) \neq 0$ and the initial guess is sufficiently close to α.

SOLUTION: We can write Newton iteration in the form of (b) with

$$g(x) := x - \frac{f(x)}{f'(x)}.$$

Thus

$$g'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)f''(x)}{(f'(x))^2}.$$

Thus it follows that $g'(\alpha) = 0$ and hence there is an interval $[a, b]$, including α, and $\mu < 1$ such that

$$|g'(x)| \leq \mu \quad \forall x \in [a, b].$$

The result follows.

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